Lifting of S^1-valued maps in sums of Sobolev spaces

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ABSTRACT. We describe, in terms of lifting, the closure of smooth $\mathbb{S}^1$-valued maps in the space $W^{s,p}((-1,1)^N;\mathbb{S}^1)$. (Here, $0 < s < \infty$ and $1 \le p < \infty$.) This description follows from an estimate for the phase of smooth maps: let $0 < s < 1$, let $\varphi \in C^\infty([-1,1]^N;\mathbb{R})$ and set $u = e^{\imath\varphi}$. Then we may split $\varphi = \varphi_1 + \varphi_2$, where the smooth maps φ_1 and φ_2 satisfy

$$(*) \quad |\varphi_1|_{W^{s,p}} \le C|u|_{W^{s,p}} \quad \text{and} \quad \|\nabla\varphi_2\|_{L^{sp}}^{sp} \le C|u|_{W^{s,p}}^p.$$

$(*)$ was proved for $s = 1/2$, $p = 2$ and arbitrary space dimension N by Bourgain and Brezis [3] and for $N = 1$, $p > 1$ and $s = 1/p$ by Nguyen [14].
Our proof is a sort of continuous version of the Bourgain-Brezis approach (based on paraproducts).
Estimate $(*)$ answers (and generalizes) a question of Bourgain, Brezis, and the author [5].

1. Introduction

In [4], the authors addressed the problem of lifting of $\mathbb{S}^1$-valued maps in Sobolev spaces:

$(L_{s,p})$ Given an arbitrary $u \in W^{s,p}(Q;\mathbb{S}^1)$, is there a $\varphi \in W^{s,p}(Q;\mathbb{R})$ such that $u = e^{\imath\varphi}$?

Here, $0 < s < \infty$, $1 \le p < \infty$ and $Q = (-1,1)^N$. The complete answer is [4]

SPACE DIMENSION N	SIZE OF s	SIZE OF sp	ANSWER TO $(L_{s,p})$
$N = 1$	ANY	ANY	YES
$N \ge 2$	$0 < s < 1$	$0 < sp < 1$	YES
$N \ge 2$	$0 < s < 1$	$1 \le sp < N$	NO
$N \ge 2$	$0 < s < 1$	$sp \ge N$	YES
$N \ge 2$	$s \ge 1$	$1 \le sp < 2$	NO
$N \ge 2$	$s \ge 1$	$sp \ge 2$	YES

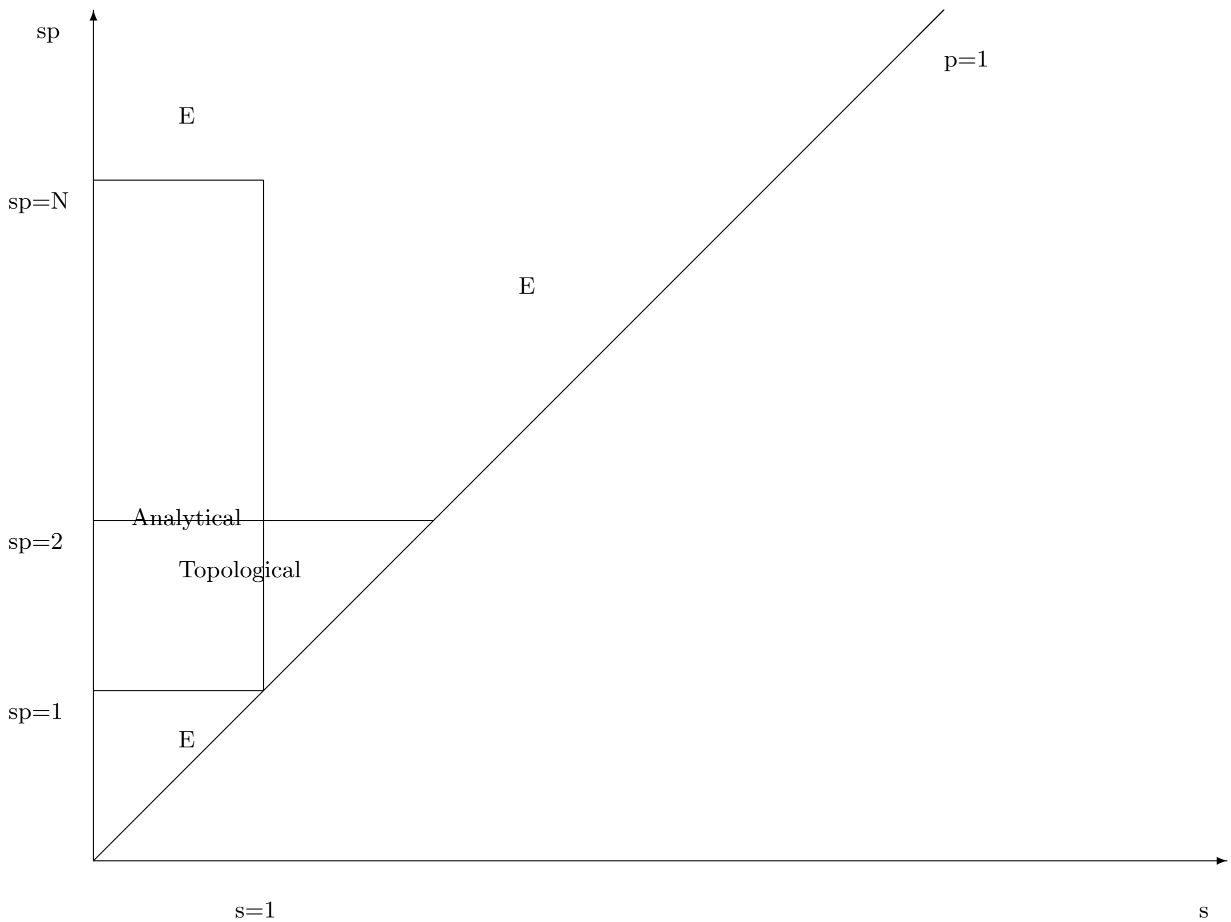
The non existence results rely on two kinds of counterexamples: **topological** and **analytical**.

Topological counterexamples. One may prove (see Proposition 1) that, if there is lifting in $W^{s,p}$, then $C^\infty(\overline{Q};\mathbb{S}^1)$ is dense in $W^{s,p}(Q;\mathbb{S}^1)$. Thus the answer to $(L_{s,p})$ is no whenever $C^\infty(\overline{Q};\mathbb{S}^1)$ is not dense in $W^{s,p}(Q;\mathbb{S}^1)$. When $1 \le sp < 2$, the typical "topological counterexample" is the

Date: June 23, 2008.

The author thanks H.-M. Nguyen for sending him the paper [14] and for stimulating discussions. He warmly thanks H. Brezis for his comments on the paper.

map $Q \ni x \mapsto \dfrac{(x_1, x_2)}{|(x_1, x_2)|}$ which belongs to $W^{s,p}(Q; \mathbb{S}^1)$ but cannot be approximated by smooth maps in the $W^{s,p}$-norm. (This goes back essentially to [16]; for a proof, see e. g. [9].)
Such a counterexample does not exist outside the "topological region" $1 \le sp < 2$. Indeed, when $sp < 1$ or $sp \ge 2$, $C^\infty(\overline{Q}; \mathbb{S}^1)$ is dense in $W^{s,p}(Q; \mathbb{S}^1)$ [9]. Thus, topological counterexamples are confined to the topological region.



In the "E" regions, there is lifting. The "topological" region is a trapezoid, the "analytical" one a rectangle

Analytical counterexamples. In the region $1 < sp < N$ and $0 < s < 1$, one may prove non existence of lifting as follows [4]: pick some $\psi \in W^{1,sp}(Q; \mathbb{R}) \setminus W^{s,p}(Q; \mathbb{R})$ (such a ψ exists, by the Sobolev "non embedding" $W^{1,sp}(Q) \not\subset W^{s,p}(Q)$). Let $u := e^{\imath\psi}$. Then $u \in W^{1,sp} \cap L^\infty$, so that $u \in W^{s,p}(Q; \mathbb{S}^1)$, by the Gagliardo-Nirenberg embedding $W^{1,sp} \cap L^\infty \subset W^{s,p}$.
This u does not lift as $u = e^{\imath\varphi}$ with $\varphi \in W^{s,p}(Q; \mathbb{R})$. Argue by contradiction: since $e^{\imath(\varphi - \psi)} = 1$, we have $\eta := \varphi - \psi \in (W^{1,sp} + W^{s,p})(Q; 2\pi\mathbb{Z})$. Thus η is constant a. e. [4] (this uses $sp \ge 1$), so

that $\psi \in W^{s,p}$, contradiction.
When $sp = 1$ and $0 < s < 1$, we still have non lifting. However, the above argument has to be slightly modified: one has to construct explicitely some $\psi \in W^{1,1}(Q;\mathbb{R}) \setminus W^{s,p}(Q;\mathbb{R})$ and $e^{\imath\psi} \in W^{s,p}(Q;\mathbb{S}^1)$ ($\psi(x) = |x|^{-\alpha}$ for appropriate $\alpha > 0$ will do it). Note that, when $sp = 1$, the property $\psi \in W^{1,1}$ does not imply $u \in W^{s,p}$; this follows from the Gagliardo-Nirenberg "non embedding" $W^{1,1} \cap L^\infty \not\subset W^{s,p}$. (Note the contrast with the embedding $W^{1,sp} \cap L^\infty \subset W^{s,p}$, valid when $sp > 1$.)
This argument implies non existence of lifting in the "analytical region" $1 \le sp < N$, $0 < s < 1$.

Unlike the topological counterexamples, the "analytical" ones belong to the subspace

$$X^{s,p} := \overline{C^\infty(\overline{Q};\mathbb{S}^1)}^{W^{s,p}}$$

(see Proposition 2).
As we have already seen, in the region "analytical\topological" there are no topological counterexamples, since $C^\infty(\overline{Q};\mathbb{S}^1)$ is dense in $W^{s,p}(Q;\mathbb{S}^1)$ when $0 < s < 1$ and $sp \ge 2$. On the other hand, there are no analytical counterexamples (=maps in $X^{s,p}$ that do not lift) in the "topological\analytical" region. Indeed, if $s \ge 1$ and $u \in X^{s,p}$, then u has a lifting in $W^{s,p}(Q;\mathbb{R})$ (Proposition 3).

The main purpose of the present paper is to prove that the analytical counterexamples we presented above are the only ones.

Theorem 1. *Assume that $0 < s < 1$, $sp \ge 1$ and that $u \in X^{s,p}$. Then $u = e^{\imath\varphi}$ for some $\varphi \in W^{s,p}(Q;\mathbb{R}) + W^{1,sp}(Q;\mathbb{R})$.*

This relies essentially on the following estimate for the lifting of smooth maps.

Theorem 2. *Let $0 < s < 1$ and $1 \le p < \infty$. Let $\varphi \in C^\infty(\overline{Q};\mathbb{R})$ and set $u := e^{\imath\varphi}$. Then we may split $\varphi = \varphi_1 + \varphi_2$, where the maps $\varphi_j \in C^\infty(Q;\mathbb{R})$, $j = 1, 2$, satisfy*

$$|\varphi_1|_{W^{s,p}} \le C|u|_{W^{s,p}} \tag{1}$$

and

$$\|\nabla\varphi_2\|_{L^{sp}}^{sp} \le C|u|_{W^{s,p}}^p. \tag{2}$$

Here, $|\ |_{W^{s,p}}$ stands for the Gagliardo sem-norm

$$|u|_{W^{s,p}} = \left(\iint_{Q\times Q} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\, dx\, dy \right)^{1/p}$$
$$\sim \left(\sum_{j=1}^N \int_{(-1,1)^{N+1}} \frac{\left| u\left(\sum_{k\ne j} x_k e_k + te_j\right) - u\left(\sum_{k\ne j} x_k e_k + se_j\right)\right|^p}{|t-s|^{1+sp}} \otimes_{k\ne j} dx_k\, dt\, ds \right)^{1/p}.$$

Two special cases of Theorem 2 were already known. In [3], Bourgain and Brezis established Theorem 2 when $s = 1/2$, $p = 2$ and N is arbitrary. Their proof adapts to the case $1 < p \le 2$,

$s = 1/p$ (and arbitrary N).
In [14], Nguyen proved Theorem 2 when $N = 1$, $p > 1$ and $s = 1/p$ (without the restriction $p \le 2$). The argument there adapts to the case where N is arbitrary, provided $sp > 1$.
Thus the really new cases are a) $sp < 1$ and b) $N \ge 2$, $sp = 1$, $p > 2$.

Theorem 1 was conjectured in [5], [12]. The results presented here were announced in [13].

The paper is organized as follows: in Section 2, we explain how lifting and density are related. In Section 3, we explain the proof of Theorem 2 and why this proof is a cousin of the Bourgain-Brezis argument. In Section 4, we establish the main estimates needed in the proof of Theorem 2. The proofs of Theorems 1 and 2 are presented in Section 5. Finally, in Section 6, we characterize $X^{s,p}$ in terms of lifting.

2. Density vs lifting

In this section, we discuss the connection between density of $C^\infty(\overline{Q};\mathbb{S}^1)$ in $W^{s,p}(Q;\mathbb{S}^1)$ and existence of lifting.

Proposition 1. *Assume that the answer to $(L_{s,p})$ is yes. Then $C^\infty(\overline{Q};\mathbb{S}^1)$ is dense in $W^{s,p}(Q;\mathbb{S}^1)$.*

Proof. Let first $s \le 1$. Write an arbitrary $u \in W^{s,p}(Q;\mathbb{S}^1)$ as $u = e^{\imath\varphi}$ with $\varphi \in W^{s,p}(Q;\mathbb{R})$. Since the map $W^{s,p}(Q;\mathbb{R}) \ni \varphi \mapsto e^{\imath\varphi} \in W^{s,p}(Q;\mathbb{S}^1)$ is clearly continuous, the conclusion follows by approximating φ with smooth maps.
Let now $s > 1$. If $u \in W^{s,p}(Q;\mathbb{S}^1)$ and if $\varphi \in W^{s,p}(Q;\mathbb{R})$ is a lifting of u, then actually φ belongs also to $W^{1,sp}(Q;\mathbb{R})$ [4]. We conclude by using the fact that the map $W^{1,sp}(Q;\mathbb{R}) \cap W^{s,p}(Q;\mathbb{R}) \ni \varphi \mapsto e^{\imath\varphi} \in W^{s,p}(Q;\mathbb{S}^1)$ is continuous [7, 11]. $\square$

As a byproduct of the proof, we obtain the following

Corollary 1. *Asume that $u \in W^{s,p}(Q;\mathbb{S}^1)$ has a lifting in $W^{s,p}(Q;\mathbb{R})$. Then u is in $X^{s,p}$.*

Proposition 2. *Assume that $0 < s < 1$ and $sp \ge 1$. Let $\psi \in W^{1,sp}(Q;\mathbb{R})$ and set $u = e^{\imath\psi}$. Then*
a) When $sp > 1$, we have $u \in X^{s,p}$.
b) When $sp = 1$ and $u \in W^{s,p}(Q;\mathbb{S}^1)$, we have $u \in X^{s,p}$.

Proof. a) The mapping $W^{1,sp}(Q;\mathbb{R}) \ni \psi \mapsto e^{\imath\psi} \in W^{s,p}(Q;\mathbb{S}^1)$ being continuous (this is an easy consequence of the Gagliardo-Nirenberg inclusion $W^{1,sp} \cap L^\infty \subset W^{s,p}$), the conclusion is immediate.
b) We may assume that $N \ge 2$, for otherwise $X^{s,p} = W^{s,p}(Q;\mathbb{S}^1)$. The main ingredient we use in the proof is the approximation technique for $W^{s,p}$-maps (when $0 < s < 1$ and $sp \le N$) in [9], which is recalled below.
We first extend u by reflections accross ∂Q. Since $0 < s < 1$, this procedure will yield a map in $W^{s,p}((-2,2)^N;\mathbb{S}^1)$. We next extend this map to a map in $W^{s,p}(\mathbb{R}^N;\mathbb{R}^2)$. We finally obtain a map, still denoted u, which is in $W^{s,p}(\mathbb{R}^N)$ and is, in addition, $\mathbb{S}^1$-valued in a neighborhood of $\overline{Q}$. Moreover, u has, in a neighborhood of $\overline{Q}$, a $W^{1,1}$-lifting (still denoted ψ).
We next explain how to approximate maps as above (which are $W^{s,p}$, $\mathbb{S}^1$-valued and with a $W^{1,1}$-lifting near $\overline{Q}$) by maps with a simple structure. To each $\varepsilon > 0$ and $T \in \mathbb{R}^N$ we may associate

a unique grid $\mathscr{G} = \mathscr{G}_{T,\varepsilon}$ of size 2ε passing through T, namely $\mathscr{G} = \bigcup_{m\in\mathbb{Z}^N} (T + (0,2\varepsilon)^N))$. For $j = 0,\dots,N$, one may define the j^{th}-dimensional skeleton $\mathscr{C}^j = \mathscr{C}^j_{T,\varepsilon}$ of $\mathscr{G}$ as follows: $\mathscr{C}^N$ is $\mathbb{R}^N$, i. e., the union of the cubes in $\mathscr{G}$. $\mathscr{C}^{N-1}$ is the union of the faces of the cubes in $\mathscr{G}$. By backward induction, $\mathscr{C}^j$ is the union of the (geometrical) boundaries of the $(j+1)$-dimensional faces that form $\mathscr{C}^{j+1}$.
Let $u : \mathbb{R}^N \to \mathbb{R}$. To each ε, T and j we may associate a map $u_{\varepsilon,T,j} : \mathbb{R}^N \to \mathbb{R}^M$ as follows: let $g = g_{T,\varepsilon,j}$ be the restriction of u to $\mathscr{C}^j = \mathscr{C}^j_{T,\varepsilon}$. We extend g to $\mathscr{C}^{j+1}$ "homogenously". More specifically, let C be the center of some face F of $\mathscr{C}^{j+1}$. Then the homogeneous extension of g from ∂F to F is the map which is constant on segments joining C to points of ∂F. Equivalently, the desired extension is $H_{j+1}g$, where $H_{j+1}g(x) = g\left(\dfrac{\varepsilon}{\|x-C\|_\infty}(x-C)\right)$ if $x \in F$.
So far, we have a map $H_{j+1}g$ defined on $\mathscr{C}^{j+1}$. We extend it homogeneously first to $\mathscr{C}^{j+2}$, next to $\mathscr{C}^{j+3}$ and so on. We end up with $u_{\varepsilon,T,j} := H_N \circ H_{N-1} \circ \ldots \circ H_{j+1}g$.
We may know state the main result in [9]:
Assume that $u \in W^{s,p}(\mathbb{R}^N)$, with $0 < s < 1$ and $sp < N$. Set $j = [sp]$ (the largest integer not exceeding sp). Then there is a sequence $\varepsilon_k \to 0$ and, for each k, there is a full measure set $A_k \subset \mathbb{R}^N$ such that $u_{\varepsilon_k,T_k,j} \to u$ in $W^{s,p}$ whenever $T_k \in A_k$. [1]
We now return to the proof of b).
The set A_k being of full measure, we may assume that the T_k's have been chosen such that:
(i) $u_{|\mathcal{C}^1} \in W^{s,p}$;
(ii) there is a cube Q_k containing $\overline{Q}$ such that $\psi_{|\mathcal{C}^1\cap Q_k} \in W^{1,1}$. In addition, we may assume that Q_k is union of some cubes in $\mathcal{G}$.
Let now B be the boundary of a square $S \in \mathcal{C}^2 \cap Q_k$. Since $u_{|B} \in W^{s,p}$, we may lift, locally on B, $u = e^{\imath\varphi}$ with $\varphi \in W^{s,p}$. (Recall that in one dimension, lifting always exists.) Since $2\pi\mathbb{Z}$-valued maps in $W^{s,p} + W^{1,1}$ are locally constant [4], we find that $\psi_{|B} \in W^{s,p}$. Thus, restricted to $\mathcal{C}^1 \cap Q_k$, u has a lifting in $W^{s,p}$.
It is easy to see that, if $f \in W^{\sigma,q}(\mathcal{C}^l)$, where $0 < \sigma < 1$ and $\sigma q < l + 1$, then $H_{l+1}f \in W^{\sigma,q}(\mathcal{C}^{l+1})$ [9]. Applying this property with $\sigma = s$, $q = p$ and $l = 2,\dots,N$, we find that, in Q, we have $u_{\varepsilon_k,T_k,1} = e^{\imath\psi_k}$, where $\psi_k \in W^{s,p}(Q;\mathbb{R})$. Since clearly $e^{\imath\psi_k} \in X^{s,p}$, we find that $u \in X^{s,p}$. □

Proposition 3. *Let $s \geq 1$. If $u \in X^{s,p}$, then u lifts in $W^{s,p}(Q;\mathbb{R})$.*

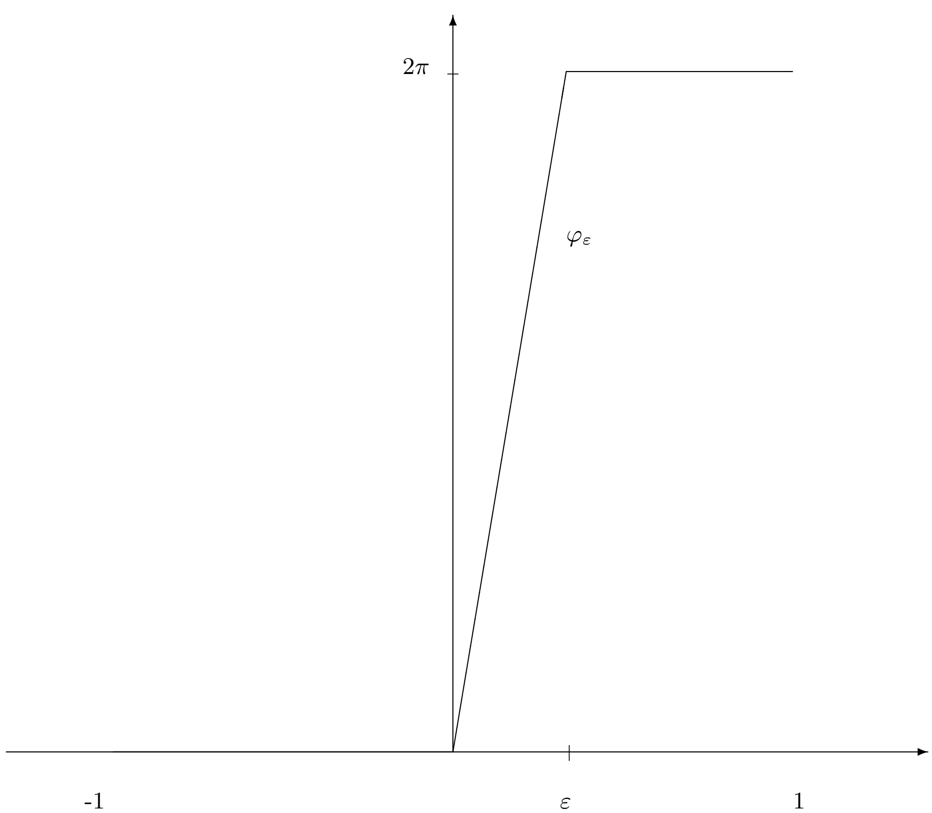
Proof. When $s \geq 1$, a map $u = u_1 + \imath u_2 \in W^{s,p}(Q;\mathbb{S}^1)$ has a lifting in $W^{s,p}(Q;\mathbb{R})$ if and only if the vector field $Y := u_1\nabla u_2 - u_2\nabla u_1$ is closed in the distribution sense, i. e. if (*) $\dfrac{\partial Y_j}{\partial x_k} = \dfrac{\partial Y_k}{\partial x_j}$ in $\mathcal{D}'(Q)$, $j,k = 1,\dots,N$ [4].
When $u = e^{\imath\varphi}$ with $\varphi \in C^2$, (*) becomes $\dfrac{\partial^2\varphi}{\partial x_j \partial x_k} = \dfrac{\partial^2\varphi}{\partial x_k \partial x_j}$ and is clearly satisfied. Since the

[1]In the proof of Proposition 2, we will apply this approximation result to the $\mathbb{R}^2$-valued map u when $s = 1/p$ (and thus $j = 1$).

mapping $W^{s,p}(Q;\mathbb{S}^1) \ni u \mapsto Y \in L^1(Q) \subset \mathcal{D}'(Q)$ is clearly continuous, we find that (*) still holds for $u \in X^{s,p}$. $\square$

3. Heuristics of the proof of Theorem 2

Recall that, in one dimension, lifting always exists. One may thus hope that, for each s and p and for each $u \in W^{s,p}((-1,1);\mathbb{S}^1)$, one may find a lifting $\psi \in W^{s,p}((-1,1);\mathbb{R})$ of u satisfying in addition an estimate of the form $\|\varphi\|_{W^{s,p}} \le F(\|u\|_{W^{s,p}})$. This is indeed true except when $sp = 1$ [4].
[There is a parallel between this situation and the case of lifting in classes of smooth maps. It is easy to see that, when $u \in C^k([-1,1];\mathbb{S}^1)$ for some $k \ge 1$, the derivatives of the smooth lifting φ of u are controlled by those of u. However, when $k = 0$, the uniform norm of u is always 1, while the one of φ is arbitrary, so that there is no control of φ in terms of u.
It is thus not a surprise that there is no control in $W^{1/p,p}$, which is the space that "almost" embeds in C^0.]
Indeed, let φ_ε be as in the picture below:



Unlike its phase φ_ε, $e^{\imath\varphi_\varepsilon}$ remains bounded in $W^{1/p,p}((-1,1))$ $(p > 1)$

Since $\varphi_\varepsilon \to \chi_{(0,1]}$ and non constant step functions are not in $W^{1/p,p}$, it follows that $\|\varphi_\varepsilon\|_{W^{1/p,p}} \to \infty$ as $\varepsilon \to 0$.
Consider now $u_\varepsilon := e^{\imath\varphi_\varepsilon}$. It is easy to see that $u_1 \in W^{1/p,p}(\mathbb{R})$ (while φ_1 belongs only to $W^{1/p,p}_{loc}(\mathbb{R})$, but not to $W^{1/p,p}(\mathbb{R})$). Since $u_\varepsilon(\cdot) = u_1(\cdot/\varepsilon)$ and the $W^{1/p,p}$-semi-norm is scale invariant in $\mathbb{R}$, it follows that u_ε remains bounded in $W^{1/p,p}$ as $\varepsilon \to 0$.
[This does not contradict Theorem 2, since φ_ε is bounded in $W^{1,1}$.]
This is a typical situation where one needs the $W^{1,sp}$-part, φ_2, of φ: the case where u is strongly oscillating. In contrast, the $W^{s,p}$-part, φ_1, is controlled by u provided u has small oscillations.
In practice, we obtain the decomposition $\varphi = \varphi_1 + \varphi_2$ as follows: we will derive a formula for the lifting φ of maps with small amplitude oscillations. When u is arbitrary, this formula will be used to define φ_1. We next simply set $\varphi_2 := \varphi - \varphi_1$.
In order to derive the formula of φ_1, assume that $u \in W^{s,p}(Q;\mathbb{S}^1)$ is close to the constant 1. Assume for simplicity that u has been extended to $\mathbb{R}^N$ as an $\mathbb{S}^1$-valued map that equals 1 at infinity; we still denote by u this extension. Let $v = v(x,\varepsilon)$ be an extension by averages of u to $\mathbb{R}^N \times \mathbb{R}_+$ (thus $v(x,\varepsilon) = u * \rho_\varepsilon(x)$, where ρ is a suitable mollifier). Since u is close to 1, v is also close to 1. In particular, v is far away from 0, so that $w := v/|v|$ is as smooth as v.
We may write (at least when $\varepsilon > 0$) $w = e^{\imath\psi}$ for some smooth ψ. Since u is 1 at infinity, we have $\lim\limits_{\varepsilon\to\infty} w(x,\varepsilon) = 1$, which suggests that we may pick ψ such that $\lim\limits_{\varepsilon\to\infty} \psi(x,\varepsilon) = 0$. This allows to write formally $u(x) = e^{\imath\varphi(x)}$, where $\varphi(x) := \psi(x,0)$ and

$$\varphi(x) = -\psi(x,\varepsilon)\Big|_{\varepsilon=0}^{\varepsilon=\infty} = -\int_0^\infty \frac{\partial\psi}{\partial\varepsilon}(x,\varepsilon)\,d\varepsilon = -\int_0^\infty w(x,\varepsilon)\wedge\frac{\partial w}{\partial\varepsilon}(x,\varepsilon)\,d\varepsilon.$$

It turns out that this is the right formula φ_1, provided we choose a more convenient w.

We may now give the explicit splitting $\varphi = \varphi_1 + \varphi_2$ in the proof of Theorem 2. Let $\varphi \in C^\infty(\overline{Q};\mathbb{R})$ and set $u = e^{\imath\varphi}$. Then we may extend u to some compactly supported map in $W^{s,p}(\mathbb{R}^N)$, still denoted u, such that $|u| \le 3$ and $|u|_{W^{s,p}(\mathbb{R}^N)} \le C|u|_{W^{s,p}(\mathbb{R}^N)}$ (Lemma 8). Let $\rho \in C_0^\infty$ be a mollifier (whose precise properties will be specified in Section 4). Let $v(x,\varepsilon) = u * \rho_\varepsilon(x)$.
Assume first that $W^{s,p}\cap L^\infty$ is contained in $W^{1-1/2sp,2sp}$, i. e. that $s \ge 1 - \dfrac{1}{2sp}$ and $sp \le 1$.[2] Then Theorem 2 works with

$$\varphi_1(x) = -\int_0^\infty v(x,\varepsilon)\wedge\frac{\partial v}{\partial\varepsilon}(x,\varepsilon)\,d\varepsilon, \quad \varphi_2 = \varphi - \varphi_1. \tag{3}$$

In general (i. e., when we do not assume that $W^{s,p}\cap L^\infty$ is contained in $W^{1-1/2sp,2sp}$), one has to project v on $\mathbb{S}^1$. More specifically, let $\Pi \in C^\infty(\mathbb{R}^2;\mathbb{R}^2)$ be such that $\Pi(z) = z/|z|$ near $\mathbb{S}^1$ and set $w := \Pi(v)$. Then we may choose, in the proof of Theorem 2,

$$\varphi_1(x) = -\int_0^\infty w(x,\varepsilon)\wedge\frac{\partial w}{\partial\varepsilon}(x,\varepsilon)\,d\varepsilon, \quad \varphi_2 = \varphi - \varphi_1. \tag{4}$$

[2]This covers the case $s = 1/2, p = 2$ treated in [3].

We end this section by comparing our approach to the Bourgain-Brezis one [3].
The fact that φ_1 given by (3) belongs to $W^{s,p}$ is reminiscent from standard estimates on paraproducts (for a quick introduction, see, e. g., [10]). Recall that (for a suitable mollifier ρ) the Littlewood-Paley decomposition of a function f is $f = \sum_{j\geq 0} LP_j(f)$, where $LP_0(f) = f * \rho$ and, for $j \geq 1$, $LP_j(f) = f * \rho_{2^{-j}} - f * \rho_{2^{-j+1}}$. Recall also that $W^{s,p} \cap L^\infty$ is an algebra. Thus, if $f, g \in W^{s,p} \cap L^\infty$, then $\sum_{j,k} LP_j(f) LP_k(g) \in W^{s,p}$. The paraproducts technique yields slightly more:

$$\text{each of the sums } \sum_{j\leq k} LP_j(f)LP_k(g) \text{ and } \sum_{j>k} LP_j(f)LP_k(g) \text{ is in } W^{s,p}. \tag{5}$$

It is well-known to the experts (though difficult to find in the literature) that each "Littlewood-Paley like" (i. e., via sequences) property of some functions space has continuous analogues.[3] These analogues are obtained by replacing $LP_j(f)$ by an integral, e. g.

$$f = LP_0(f) + \sum_{j=-\infty}^{\infty} (f * \rho_{2^{-j}} - f * \rho_{2^{-j+1}}) = LP_0(f) - \int_0^1 f * \frac{\partial}{\partial\varepsilon}(\rho_\varepsilon)\, d\varepsilon;$$

another used decomposition is $f = -\int_0^\infty f * \frac{\partial}{\partial\varepsilon}(\rho_\varepsilon)\, d\varepsilon$. Equivalently, if $F(x,\varepsilon) = f * \rho_\varepsilon(x)$ is the extension of f to $\mathbb{R}^N \times \mathbb{R}_+$, then $f(x) = -\int_0^\infty \frac{\partial F}{\partial\varepsilon}(x,\varepsilon)\, d\varepsilon$.
With F, G the extensions of $f, g \in W^{s,p} \cap L^\infty$, an analogue of (5) is

$$x \mapsto \int_0^\infty F(x,\varepsilon)\frac{\partial G}{\partial\varepsilon}(x,\varepsilon)\, d\varepsilon \in W^{s,p}, \quad \forall\ f, g \in W^{s,p} \cap L^\infty. \tag{6}$$

Property (6) is true[4] and implies that the function φ_1 given by (3) is in $W^{s,p}$. The same conclusion holds for the φ_1 given by (4), but the argument is more involved.
We may now compare our construction to the Bourgain-Brezis one: their splitting is $\varphi_1 = \sum_{j\leq k} LP_j(u) \wedge LP_k(u)$, $\varphi_2 = \varphi - \varphi_1$. This is nothing else than a discrete analogue of (3). However, it seems difficult to cover the case $sp \neq 1$ using this decomposition.

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[3]Example #1: one may define a square function of an L^p-function f defined in $\mathbb{R}^N$ either through the formula $S_1 f(x) := \left(\sum |LP_j f(x)|^2\right)^{1/2}$, or through $S_2 f(x) := \left(\int_0^\infty (f * \rho_\varepsilon(x))^2 \frac{d\,\varepsilon}{\varepsilon}\right)^{1/2}$, for appropriate ρ. In both cases, we have the "square function theorem" of Littlewood and Paley $S_j(f) \sim \|f||_{L^p}$, $1 < p < \infty$ [17] II. 6.
Example #2: functions f in Triebel-Lizorkin spaces can be characterized both in terms of the Littlewood-Paley decomposition of f and in terms of the behavior of the solution of the heat equation with initial condition f [18] 2.12.

[4]This is presumably well-known. For another ρ, (6) is nothing else but the conclusion of Lemma 19.

4. Estimates for extensions by averages

Throughout this section, $u \in \mathrm{Lip}(\mathbb{R}^N)$ is such that $|u| \le 3$ and u is constant outside some compact[5]. We set, for $\varepsilon > 0$, $v(x,\varepsilon) = u * \rho_\varepsilon(x)$.
We assume that the mollifier ρ satisfies[6]:

$$\rho \in C_0^\infty(\mathbb{R}^N), \quad \rho \ge 0, \quad \operatorname{supp} \rho \subset B(0,2), \quad \rho = 0 \text{ in } B(0,1).$$

C will denote a constant depending on s, p, N, but not on u (provided $|u| \le 3$).

Lemma 1. *Assume that* $0 < s < 1$, $1 \le p < \infty$. *Then*

$$\int_{\mathbb{R}^N} \int_0^\infty \varepsilon^{p-sp-1} |Dv(x,\varepsilon)|^p \, d\varepsilon \, dx \le C |u|^p_{W^{s,p}}. \tag{7}$$

Proof. Set $\zeta_j := \partial_j \rho$, $j = 1, \dots, N$, and $\zeta_0 := -\sum_{j=1}^N \partial_j (x_j \rho)$. It is easy to see that

$$\frac{\partial}{\partial x_j} v(x,\varepsilon) = \frac{1}{\varepsilon} u * (\zeta_j)_\varepsilon(x) \quad \text{and} \quad \frac{\partial}{\partial \varepsilon} v(x,\varepsilon) = \frac{1}{\varepsilon} u * (\zeta_0)_\varepsilon(x). \tag{8}$$

Noting that each ζ_j is supported in $B(0,2)$ and has zero integral, we find that (7) is a consequence of

$$\int_{\mathbb{R}^N} \int_0^\infty \varepsilon^{-sp-1} |u * \zeta_\varepsilon(x)|^p \, d\varepsilon \, dx \le C(\zeta) |u|^p_{W^{s,p}}, \quad \forall\, \zeta \in C_0^\infty(B(0,2)) \text{ such that } \int \zeta = 0. \tag{9}$$

In order to prove (9), we note that

$$|u * \zeta_\varepsilon(x)|^p = \left| \frac{1}{\varepsilon^N} \int_{B(0,2\varepsilon)} (u(x-y) - u(x)) \zeta\left(\frac{y}{\varepsilon}\right) dy \right|^p \le \frac{C}{\varepsilon^N} \int_{B(0,2\varepsilon)} |u(x-y) - u(x)|^p \, dy.$$

Thus

$$\begin{aligned} \int_{\mathbb{R}^N} \int_0^\infty \varepsilon^{-sp-1} |u * \zeta_\varepsilon(x)|^p \, d\varepsilon \, dx \le & C \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \int_{|y|/2}^\infty \frac{1}{\varepsilon^{N+sp+1}} \, d\varepsilon \, |u(x-y) - u(x)|^p \, dy \, dx \\ = & C \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} \, dx \, dy = C |u|^p_{W^{s,p}}. \end{aligned}$$

□

Lemma 2. *Assume that* $0 < s < 1$, $1 \le p < \infty$. *Then*

$$I := \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{1}{|x-y|^{N+sp}} \left(\int_0^{|x-y|} |Dv(x,\varepsilon)| \, d\varepsilon \right)^p dx \, dy \le C |u|^p_{W^{s,p}}. \tag{10}$$

[5]The second assumption is used only in Lemmas 5-7.

[6]Assumptions on ρ are not crucial. The results in this section are true for any reasonable mollifier. However, our assumptions make the proofs simpler.

Proof. Let α be such that $\frac{1}{p}-1<\alpha<\frac{1}{p}-1+s$. We first note that, for $r>0$ and $p>1$, we have

$$\begin{aligned}\left(\int_0^r |Dv(x,\varepsilon)|\,d\varepsilon\right)^p =&\left(\int_0^r \frac{|Dv(x,\varepsilon)|}{\varepsilon^\alpha}\varepsilon^\alpha\,d\varepsilon\right)^p \le \int_0^r \frac{|Dv(x,\varepsilon)|^p}{\varepsilon^{\alpha p}}\,d\varepsilon\left(\int_0^r \varepsilon^{\alpha p/(p-1)}\,d\varepsilon\right)^{p-1}\\ =&Cr^{\alpha p+p-1}\int_0^r \frac{|Dv(x,\varepsilon)|^p}{\varepsilon^{\alpha p}}\,d\varepsilon\end{aligned} \tag{11}$$

(here we use the fact that $\frac{\alpha p}{p-1}>-1$, which is equivalent to $\alpha>\frac{1}{p}-1$). On the other hand, it is immediate that the conclusion of (11) still holds when $p=1$.
If we write $y=x+r\omega$, with $r=|x-y|$ and $\omega\in\mathbb{S}^{N-1}$ and use (11) with $r=|x-y|$, we find that

$$\begin{aligned}I=&C\int_{\mathbb{R}^N}\int_{\mathbb{S}^{N-1}}\int_0^\infty \frac{1}{r^{sp+1}}\left(\int_0^r |Dv(x,\varepsilon)|\,d\varepsilon\right)^p dr\,d\omega\,dx\\ \le&C\int_{\mathbb{R}^N}\int_{\mathbb{S}^{N-1}}\int_0^\infty r^{\alpha p+p-sp-2}\int_0^r \frac{|Dv(x,\varepsilon)|^p}{\varepsilon^{\alpha p}}\,d\varepsilon\,dr\,d\omega\,dx\\ =&C\int_{\mathbb{R}^N}\int_0^\infty \frac{|Dv(x,\varepsilon)|^p}{\varepsilon^{\alpha p}}\int_\varepsilon^\infty r^{\alpha p+p-sp-2}\,dr\,d\varepsilon\,dx\\ =&C\int_{\mathbb{R}^N}\int_0^\infty \varepsilon^{p-sp-1}|Dv(x,\varepsilon)|^p\,d\varepsilon\,dx\le C|u|^p_{W^{s,p}}.\end{aligned}$$

Here, we rely on the inequality $\alpha p+p-sp-2<-1$ (which amounts to $\alpha<\frac{1}{p}-1+s$) and on Lemma 1. $\square$

Lemma 3. *Assume that* $0<s<1$, $1\le p<\infty$. *Then*

$$J:=\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{1}{|x-y|^{N+sp}}\left(\int_{|x-y|}^\infty |Dv(x,\varepsilon)-Dv(y,\varepsilon)|\,d\varepsilon\right)^p dx\,dy\le C|u|^p_{W^{s,p}}. \tag{12}$$

Proof. In view of (8), it suffices to prove that

$$\tilde J:=\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{1}{|x-y|^{N+sp}}\left(\int_{|x-y|}^\infty \frac{|u*\zeta_\varepsilon(x)-u*\zeta_\varepsilon(y)|}{\varepsilon}\,d\varepsilon\right)^p dx\,dy\le C|u|^p_{W^{s,p}} \tag{13}$$

under the assumptions $\zeta\in C_0^\infty(B(0,2))$, $\int\zeta=0$ and $\zeta=0$ in $B(0,1)$.
If we set $\Phi:=D\zeta$, then $D_x(u*\zeta_\varepsilon)=\frac{1}{\varepsilon}u*\Phi_\varepsilon$, $\Phi\in C_0^\infty(B(0,2))$, $\int\Phi=0$ and $\Phi=0$ in $B(0,1)$.
We find that

$$|D_x(u*\zeta_\varepsilon)(x)|=\frac{1}{\varepsilon^{N+1}}\left|\int_{B(0,2\varepsilon)\setminus B(0,\varepsilon)}(u(x+z)-u(x))\Phi\left(\frac{z}{\varepsilon}\right)dz\right|\le\frac{C}{\varepsilon^{N+1}}\int_{\varepsilon\le|z|\le2\varepsilon}|u(x+z)-u(x)|\,dz.$$

Therefore, with $r := |y-x|$ and $\omega := \dfrac{y-x}{r}$, we have

$$\begin{aligned}|u*\zeta_\varepsilon(x)-u*\zeta_\varepsilon(y)| =& r\left|\int_0^1 D_x(u*\zeta_\varepsilon)(x+rt\omega)\cdot\omega\,dt\right|\\ \le& \frac{Cr}{\varepsilon^{N+1}}\int_0^1\int_{\varepsilon\le|z|\le2\varepsilon}|u(x+rt\omega+z)-u(x+rt\omega)|\,dz\,dt,\end{aligned}$$

which implies that

$$\begin{aligned}\int_{|x-y|}^\infty \frac{|u*\zeta_\varepsilon(x)-u*\zeta_\varepsilon(y)|}{\varepsilon}\,d\varepsilon \le& Cr\int_{|z|\ge r}\int_0^1\int_{|z|/2}^{|z|}\frac{|u(x+rt\omega+z)-u(x+rt\omega)|}{\varepsilon^{N+2}}\,d\varepsilon\,dt\,dz\\ =& Cr\int_0^1\int_{|z|\ge r}\frac{|u(x+rt\omega+z)-u(x+rt\omega)|}{|z|^{N+1}}\,dz\,dt.\end{aligned}\tag{14}$$

Let now α be such that $\dfrac{N}{p}-N+s-1<\alpha<\dfrac{N}{p}-N$. We perform the following calculation only for $p>1$, but the reader may easily see that its conclusion still holds for $p=1$. Using (14), we find that

$$\begin{aligned}K :=& \left(\int_{|x-y|}^\infty \frac{|u*\zeta_\varepsilon(x)-u*\zeta_\varepsilon(y)|}{\varepsilon}\,d\varepsilon\right)^p\\ \le& Cr^p\left(\int_0^1\int_{|z|\ge r}\frac{|u(x+rt\omega+z)-u(x+rt\omega)|}{|z|^{N+1+\alpha}}|z|^\alpha\,dz\,dt\right)^p\\ \le& Cr^p\int_0^1\int_{|z|\ge r}\frac{|u(x+rt\omega+z)-u(x+rt\omega)|^p}{|z|^{Np+p+\alpha p}}\,dz\,dt\left(\int_{|z|\ge r}|z|^{\alpha p/(p-1)}\,dz\right)^{p-1}\\ =& Cr^{p+\alpha p+Np-N}\int_0^1\int_{|z|\ge r}\frac{|u(x+rt\omega+z)-u(x+rt\omega)|^p}{|z|^{Np+p+\alpha p}}\,dz\,dt\end{aligned}\tag{15}$$

since $\dfrac{\alpha p}{p-1}<-N$ (equivalently, since $\alpha<\dfrac{N}{p}-N$). Inserting (15) into the definition of $\tilde J$ and computing the integral in y in spherical coordinates, we obtain

$$\begin{aligned}\tilde J \le& C\int_{\mathbb{R}^N}\int_{\mathbb{S}^{N-1}}\int_0^\infty\int_0^1\int_{|z|\ge r} r^{p+\alpha p+Np-N-sp-1}\frac{|u(x+rt\omega+z)-u(x+rt\omega)|^p}{|z|^{Np+p+\alpha p}}\,dz\,dt\,dr\,d\omega\,dx\\ =& C\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|u(x+z)-u(x)|^p}{|z|^{Np+p+\alpha p}}\int_0^{|z|} r^{p+\alpha p+Np-N-sp-1}\,dr\,dx\,dz\\ =& C\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|u(x+z)-u(x)|^p}{|z|^{N+sp}}\,dx\,dz = C|u|^p_{W^{s,p}}.\end{aligned}$$

Here, we used the inequality $p+\alpha p+Np-N-sp-1>-1$, which amounts to $\alpha>\dfrac{N}{p}-N+s-1$. $\square$

Lemma 4. *Assume $0 < s < 1$, $1 \le p < \infty$. Then*

$$L := \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{1}{|x-y|^{N+sp}} \left(\int_{|x-y|}^{\infty} \frac{|x-y|}{\varepsilon} |Dv(x,\varepsilon)| \, d\varepsilon \right)^p dx\, dy \le C|u|^p_{W^{s,p}}. \tag{16}$$

Proof. In spherical coordinates $y = x + r\omega$ we have

$$\begin{aligned} L =& C \int_{\mathbb{R}^N} \int_{\mathbb{S}^{N-1}} \int_0^\infty r^{p-sp-1} \left(\int_r^\infty \frac{|Dv(x,\varepsilon)|}{\varepsilon} \, d\varepsilon \right)^p dr\, d\omega\, dx \\ =& C \int_{\mathbb{R}^N} \int_0^\infty r^{p-sp-1} \left(\int_r^\infty \frac{|Dv(x,\varepsilon)|}{\varepsilon} \, d\varepsilon \right)^p dr\, dx, \end{aligned}$$

so that (16) amounts to

$$\tilde{L} := \int_{\mathbb{R}^N} \int_0^\infty r^{p-sp-1} \left(\int_r^\infty \frac{|Dv(x,\varepsilon)|}{\varepsilon} \, d\varepsilon \right)^p dr\, dx \le C|u|^p_{W^{s,p}}.$$

Let α be such that $\dfrac{1}{p} + s - 2 < \alpha < \dfrac{1}{p} - 1$. We perform the calculation below for $p > 1$; clearly, its conclusion still holds for $p = 1$. We have

$$\begin{aligned} \left(\int_r^\infty \frac{|Dv(x,\varepsilon)|}{\varepsilon} \, d\varepsilon \right)^p =& \left(\int_r^\infty \frac{|Dv(x,\varepsilon)|}{\varepsilon^{\alpha+1}} \varepsilon^\alpha \, d\varepsilon \right)^p \\ \le& \int_r^\infty \frac{|Dv(x,\varepsilon)|^p}{\varepsilon^{p+\alpha p}} \, d\varepsilon \left(\int_r^\infty \varepsilon^{\alpha p/(p-1)} \, d\varepsilon \right)^{p-1} \\ \le& C r^{\alpha p + p - 1} \int_r^\infty \frac{|Dv(x,\varepsilon)|^p}{\varepsilon^{p+\alpha p}} \, d\varepsilon, \end{aligned} \tag{17}$$

since $\dfrac{\alpha p}{p-1} < -1$ (i. e., $\alpha < \dfrac{1}{p} - 1$). Combining (17) with the definition of $\tilde{L}$, we obtain

$$\begin{aligned} \tilde{L} \le& C \int_{\mathbb{R}^N} \int_0^\infty \int_r^\infty |Dv(x,\varepsilon)|^p \varepsilon^{-p-\alpha p} r^{2p-sp+\alpha p-2} \, d\varepsilon\, dr\, dx \\ =& C \int_{\mathbb{R}^N} \int_0^\infty \int_0^\varepsilon |Dv(x,\varepsilon)|^p \varepsilon^{-p-\alpha p} r^{2p-sp+\alpha p-2} \, dr\, d\varepsilon\, dx \\ =& C \int_{\mathbb{R}^N} \int_0^\infty \varepsilon^{p-sp-1} |Dv(x,\varepsilon)|^p \, d\varepsilon\, dx \le C|u|^p_{W^{s,p}}. \end{aligned}$$

Here, we used the fact that $2p - sp + \alpha p - 2 > -1$ (which is equivalent to $\alpha > \dfrac{1}{p} + s - 2$) and Lemma 1. $\square$

Lemma 5. *For each $x \in \mathbb{R}^N$, the integral $\displaystyle\int_0^\infty |Dv(x,\varepsilon)| \, d\varepsilon$ is convergent.*

Proof. Since v is Lipschitz, it suffices to prove that $\int_1^\infty |Dv(x,\varepsilon)|\,d\varepsilon$ converges. With a the value of u at infinity, this follows from

$$|Dv(x,\varepsilon)| = |D[(u-a)*\rho_\varepsilon(x)]| = |(u-a)*D[\rho_\varepsilon(x)]| \le \|u-a\|_{L^1}\|D[\rho_\varepsilon(x)]\|_{L^\infty} \le \frac{C}{\varepsilon^{N+1}}.$$

□

In the same spirit, we have the following result, whose straightforward proof will be omitted

Lemma 6. *Assume u complex-valued, Lipschitz in $\mathbb{R}^N$ and smooth in Q. Set*

$$w(x) = w_\Pi(x) := \int_0^\infty \Pi\circ v(x,\varepsilon)\wedge\frac{\partial}{\partial\varepsilon}(\Pi\circ v(x,\varepsilon))\,d\varepsilon, \qquad \textit{where } \Pi\in C^\infty(\mathbb{R}^2;\mathbb{R}^2). \tag{18}$$

Then w is smooth in Q and $\partial^\alpha w(x) = \int_0^\infty \partial^\alpha\Big(\Pi\circ v(x,\varepsilon)\wedge\frac{\partial}{\partial\varepsilon}(\Pi(v(x,\varepsilon))\Big)\,d\varepsilon.$

We may now prove that the map φ_1 defined in (4) is in $W^{s,p}$ (plus norm control).

Lemma 7. *Assume that $u\in Lip(\mathbb{R}^N;\mathbb{C})$ satisfies $|u|\le 3$. Let w be defined by (18). Then*

$$|w|_{W^{s,p}} \le C|u|_{W^{s,p}}. \tag{19}$$

Proof. Set $a(x,y) := \int_0^{|x-y|} \Pi\circ v(x,\varepsilon)\wedge\frac{\partial}{\partial\varepsilon}(\Pi\circ v)(x,\varepsilon)\,d\varepsilon$ and $b(x,y) := \int_{|x-y|}^\infty \Pi\circ v(x,\varepsilon)\wedge\frac{\partial}{\partial\varepsilon}(\Pi\circ v)(x,\varepsilon)\,d\varepsilon$, so that $w(x) = a(x,y)+b(x,y)$. On the one hand, we have

$$\begin{aligned}|w(x)-w(y)| \le& |a(x,y)|+|a(y,x)|+|b(x,y)-b(y,x)|\\ \le& |b(x,y)-b(y,x)| + C\Big(\int_0^{|x-y|}|Dv(x,\varepsilon)|\,d\varepsilon + \int_0^{|x-y|}|Dv(y,\varepsilon)|\,d\varepsilon\Big).\end{aligned} \tag{20}$$

On the other hand, we have

$$\begin{aligned} b(x,y)-b(y,x) &= \int_{|x-y|}^\infty \Big(\Pi\circ v(x,\varepsilon)\wedge\frac{\partial}{\partial\varepsilon}(\Pi\circ v)(x,\varepsilon) - \Pi\circ v(y,\varepsilon)\wedge\frac{\partial}{\partial\varepsilon}(\Pi\circ v)(y,\varepsilon)\Big)\,d\varepsilon\\ &= \int_{|x-y|}^\infty (\Pi\circ v(x,\varepsilon)-\Pi\circ v(y,\varepsilon))\wedge\frac{\partial}{\partial\varepsilon}(\Pi\circ v)(x,\varepsilon)\,d\varepsilon\\ &\quad + \int_{|x-y|}^\infty \Pi\circ v(y,\varepsilon)\Big(\frac{\partial}{\partial\varepsilon}(\Pi\circ v)(x,\varepsilon) - \frac{\partial}{\partial\varepsilon}(\Pi\circ v)(y,\varepsilon)\Big)\,d\varepsilon.\end{aligned} \tag{21}$$

Since (with ζ_j as in (8))

$$|D(\Pi\circ v)(x,\varepsilon)| \le \frac{C}{\varepsilon}\sum_{j=0}^N \|u\|_{L^\infty}\|(\zeta_j)_\varepsilon\|_{L^1} \le \frac{C}{\varepsilon}, \tag{22}$$

we find that $|\Pi \circ v(x,\varepsilon) - \Pi \circ v(y,\varepsilon)| \leq \dfrac{C|x-y|}{\varepsilon}$, which yields

$$\begin{aligned}
|b(x,y)-b(y,x)| \leq & C\int_{|x-y|}^{\infty} \left|\frac{\partial}{\partial\varepsilon}(\Pi\circ v)(x,\varepsilon) - \frac{\partial}{\partial\varepsilon}(\Pi\circ v)(y,\varepsilon))\right| d\varepsilon \\
& + C\int_{|x-y|}^{\infty} \frac{|x-y|}{\varepsilon}\left|\frac{\partial}{\partial\varepsilon}(\Pi\circ v)(x,\varepsilon)\right| d\varepsilon \\
\leq & C\left(\int_{|x-y|}^{\infty} |Dv(x,\varepsilon)-Dv(y,\varepsilon)|\, d\varepsilon + \int_{|x-y|}^{\infty} \frac{|x-y|}{\varepsilon}|Dv(x,\varepsilon)|\, d\varepsilon\right).
\end{aligned} \tag{23}$$

By combining (20)-(23) to Lemmas 2-4, we find that

$$|w|^p_{W^{s,p}} \leq C(I+J+L) \leq C|u|^p_{W^{s,p}}.$$

□

5. Proof of Theorems 1 and 2

Proof of Theorem 2. Let $\varphi \in C^\infty(\overline{Q};\mathbb{S}^1)$ and set $u := e^{\imath\varphi}$.

Lemma 8. *The map u has a C^1-extension to $\mathbb{R}^N$, still denoted u, such that $|u| \leq 3$, u is constant outside $(-2,2)^N$ and $|u|_{W^{s,p}(\mathbb{R}^N)} \leq C|u|_{W^{s,p}(Q)}$.*

Proof. Let $P : W^{s,p}(Q) \to W^{s,p}(\mathbb{R}^N)$ be a linear continuous extension operator such that: P extends Lipschitz maps to Lipschitz maps, P does not increase the L^∞-norm and supp $Pv \subset (-2,2)^N$, $\forall\ v \in W^{s,p}$. Let a be the average of u on Q. It is easy to see that $\tilde{u} := a + P(u-a)$ is Lipschitz and satisfies $|\tilde{u}| \leq 3$. In addition, we have

$$|\tilde{u}|_{W^{s,p}(\mathbb{R}^N)} = |P(u-a)|_{W^{s,p}(\mathbb{R}^N)} \leq C\|u-a\|_{W^{s,p}(Q)} \leq C|u-a|_{W^{s,p}(Q)} = C|u|_{W^{s,p}(Q)};$$

we have used the Poincaré inequality $\|u-a\|_{W^{s,p}(Q)} \leq C|u-a|_{W^{s,p}(Q)}$, valid since $u-a$ has zero average. □

Let then $\Pi \in C^\infty(\mathbb{R}^2;\mathbb{R}^2)$ to be chosen later and set $\varphi_1(x) := -\displaystyle\int_0^\infty \Pi\circ v(x,\varepsilon)\wedge\frac{\partial}{\partial\varepsilon}(\Pi\circ v)(x,\varepsilon)\, d\varepsilon$.
By Lemma 19, φ_1 belongs to $W^{s,p}$ and satisfies $|\varphi_1|_{W^{s,p}} \leq C|u|_{W^{s,p}}$.
We set, for $x \in \overline{Q}$, $\varphi_2(x) := \varphi(x) - \varphi_1(x)$.

Lemma 9. *Assume that $\Pi(z) = z$, $\forall\ z \in \mathbb{S}^1$. Then*

$$D\varphi_2(x) = -2\int_0^\infty \frac{\partial}{\partial\varepsilon}(\Pi\circ v)(x,\varepsilon) \wedge D_x(\Pi\circ v)(x,\varepsilon)\, d\varepsilon. \tag{24}$$

Proof. Using Lemma 6 and the identity $D\varphi = u \wedge Du$ we have, with $w := \Pi \circ v$,

$$\begin{aligned} D\varphi_2(x) =& D\varphi(x) + \int_0^\infty D_x w(x,\varepsilon) \wedge \frac{\partial}{\partial\varepsilon} w(x,\varepsilon)\, d\varepsilon + \int_0^\infty w(x,\varepsilon) \wedge \frac{\partial}{\partial\varepsilon} D_x w(x,\varepsilon)\, d\varepsilon \\ =& w(x,\varepsilon) \wedge D_x w(x,\varepsilon)_{|\varepsilon=0} + \int_0^\infty D_x w(x,\varepsilon) \wedge \frac{\partial}{\partial\varepsilon} w(x,\varepsilon)\, d\varepsilon + w(x,\varepsilon) \wedge D_x w(x,\varepsilon) \Big|_{\varepsilon=0}^{\varepsilon=\infty} \\ & - \int_0^\infty \frac{\partial}{\partial\varepsilon} w(x,\varepsilon) \wedge D_x w(x,\varepsilon)\, d\varepsilon = -2 \int_0^\infty \frac{\partial}{\partial\varepsilon} (\Pi \circ v)(x,\varepsilon) \wedge D_x (\Pi \circ v)(x,\varepsilon)\, d\varepsilon. \end{aligned}$$

□

The remaining part of the proof of Theorem is essentially a variant of the proof of Theorem 0.1 in [5]. Up to now, the proof requires only $\Pi(u) = u$.[7] From now on, we will require that Π is an approximate projection on $\mathbb{S}^1$, e.g. we assume $\Pi(z) = \dfrac{z}{|z|}$ when $|z| \geq \dfrac{1}{2}$.

Lemma 10. *Let, for $x \in \overline{Q}$, $d(x) := \inf\{\varepsilon > 0 \ ; \ |v(x,\varepsilon)| = 1/2\}$. Then*

$$\int_Q \frac{1}{d(x)^{sp}}\, dx \leq C|u|^p_{W^{s,p}(Q)}. \tag{25}$$

Proof. Let x be such that $d(x)$ is finite. Since $|u(x) - v(x, d(x))| \geq 1/2$, we have

$$1/2 \leq |u(x) - v(x, d(x))| \leq |v(x,\cdot)|_{C^s(\mathbb{R}_+)} d(x)^s \leq C|v(x,\cdot)|_{W^{s+1/p,p}(\mathbb{R}_+)} d(x)^s,$$

so that

$$\int_Q \frac{1}{d(x)^{sp}}\, dx \leq C \int_Q |v(x,\cdot)|^p_{W^{s+1/p,p}(\mathbb{R}_+)} \leq C|v|^p_{W^{s+1/p,p}} \leq C|u|^p_{W^{s,p}(\mathbb{R}^N)} \leq C|u|^p_{W^{s,p}(Q)},$$

by the Besov lemma [2]. □

Lemma 11. *We have*

$$\int_Q |D\varphi_2|^{sp}\, dx \leq C|u|^p_{W^{s,p}(Q)}. \tag{26}$$

Proof. Set $\Omega := \{(x,\varepsilon) \ ; \ x \in Q, 0 < \varepsilon < d(x)\}$. In Ω, we have $|\Pi \circ v| \equiv 1$, so that $\dfrac{\partial}{\partial\varepsilon}(\Pi \circ v)(x,\varepsilon) \wedge D_x(\Pi \circ v)(x,\varepsilon) \equiv 0$ in Ω. In view of Lemma 9 and (22), we find that

$$\begin{aligned} \int_Q |D\varphi_2|^{sp}\, dx \leq& \int_Q \left(\int_{d(x)}^\infty |D(\Pi \circ v)(x,\varepsilon)|^2\, d\varepsilon \right)^{sp} dx \\ \leq& C \int_Q \left(\int_{d(x)}^\infty \frac{1}{\varepsilon^2}\, d\varepsilon \right)^{sp} dx \leq C \int_Q \frac{1}{d(x)^{sp}}\, dx \leq C|u|^p_{W^{s,p}(Q)}. \end{aligned}$$

□

[7] When $W^{s,p} \cap L^\infty$ is contained in $W^{1-1/2sp,2sp}$, Lemma 11 below is valid without any other restriction.

The proof of Theorem 2 is complete. □

Proof of Theorem 1. When $p > 1$ and $sp > 1$, Theorem 1 is an immediate consequence of Theorem 2 (one passes to the weak limits in (1) and (2)). Some care is needed when $p = 1$ or $sp = 1$.

Lemma 12. *Let $u \in X^{s,p}$. Then there is a sequence $\{u_k\}_{k\geq 0} \subset C^\infty(\overline{Q}; \mathbb{S}^1)$ such that $u = \prod_{k=0}^{\infty} u_k$ and $|u_k|_{W^{s,p}} \leq 2^{-k+1}|u|_{W^{s,p}}$, $\forall\ k$.*

Proof. Recall that

$$\text{if } f_j \to f \text{ in } W^{s,p}, g_j \to g \text{ in } W^{s,p}, \|f_j\|_{L^\infty} \leq C, \|g_j\|_{L^\infty} \leq C, \text{ then } f_j g_l \overset{j,l\to\infty}{\longrightarrow} fg \text{ in } W^{s,p}. \tag{27}$$

Consider a sequence $\{h_j\} \subset C^\infty(\overline{Q}; \mathbb{S}^1)$ such that $h_j \to u$. We may assume that $|h_j|_{W^{s,p}} \leq 2|u|_{W^{s,p}}$, $\forall\ j$. Using (27) with $f_j = h_j$ and $g_j = \overline{h_j}$, we find easily a sequence $j_k \to \infty$ such that $\|h_{j_{k+1}}\overline{h_{j_k}} - 1\|_{W^{s,p}} \leq 2^{-k}|u|_{W^{s,p}}$, $k \geq 0$. If we set $u_0 = h_{j_0}$ and $u_k = h_{j_k}\overline{h_{j_{k-1}}}$, $k \geq 1$, then the sequence $\{u_k\}$ has the required properties. □

In view of Theorem 2, we may write each u_k as $e^{\imath(\varphi_1^k + \varphi_2^k)}$, where $|\varphi_1^k|_{W^{s,p}} \leq C2^{-k}|u|_{W^{s,p}}$ and $\|D\varphi_2^k\|_{L^{sp}} \leq C2^{-k/s}|u|_{W^{s,p}}^{1/s}$.
Let $\varphi_1 := \sum_k \varphi_1^k$. Clearly, $\varphi_1 \in W^{s,p}$ and $|\varphi_1|_{W^{s,p}} \leq C|u|_{W^{s,p}}$. On the other hand, set $\psi := \sum_k (\varphi_2^k - \fint_Q \varphi_2^k)$, which satisfies $\psi \in W^{1,sp}$ and $\|D\psi\|_{L^{sp}}^{sp} \leq C|u|_{W^{s,p}}^p$. In addition, the map $ue^{-\imath(\varphi_1+\psi)}$ is constant. It follows that, for an appropriate $\alpha \in \mathbb{R}$, φ_1 and $\varphi_2 := \alpha + \psi$ satisfy $u = e^{\imath(\varphi_1+\varphi_2)}$ and the estimates (1)-(2).

6. Characterization of $X^{s,p}$ in terms of lifting

The results in [4], [8] and [9] give the following information about $X^{s,p}$

SPACE DIMENSION N	SIZE OF s	SIZE OF sp	DESCRIPTION OF $X^{s,p}$
ANY	$0 < s < 1$	$0 < sp < 1$	$X^{s,p} = W^{s,p}(Q;\mathbb{S}^1) = \{e^{\imath\varphi}\ ;\ \varphi \in W^{s,p}(Q;\mathbb{R})\}$
$N \geq 2$	$0 < s < 1$	$1 \leq sp < 2$	$X^{s,p} \neq W^{s,p}(Q;\mathbb{S}^1), X^{s,p} \neq \{e^{\imath\varphi}\ ;\ \varphi \in W^{s,p}(Q;\mathbb{R})\}$
$N \geq 3$	$0 < s < 1$	$2 \leq sp < N$	$X^{s,p} = W^{s,p}(Q;\mathbb{S}^1), X^{s,p} \neq \{e^{\imath\varphi}\ ;\ \varphi \in W^{s,p}(Q;\mathbb{R})\}$
ANY	$0 < s < 1$	$sp \geq N$	$X^{s,p} = W^{s,p}(Q;\mathbb{S}^1) = \{e^{\imath\varphi}\ ;\ \varphi \in W^{s,p}(Q;\mathbb{R})\}$
$N = 1$	$s \geq 1$	ANY	$X^{s,p} = W^{s,p}(Q;\mathbb{S}^1) = \{e^{\imath\varphi}\ ;\ \varphi \in W^{s,p} \cap W^{1,sp}\}$
$N \geq 2$	$s \geq 1$	$1 \leq sp < 2$	$X^{s,p} \neq W^{s,p}(Q;\mathbb{S}^1), X^{s,p} = \{e^{\imath\varphi}\ ;\ \varphi \in W^{s,p} \cap W^{1,sp}\}$
$N \geq 2$	$s \geq 1$	$sp \geq 2$	$X^{s,p} = W^{s,p}(Q;\mathbb{S}^1) = \{e^{\imath\varphi}\ ;\ \varphi \in W^{s,p} \cap W^{1,sp}\}$

The following result completes the description of $X^{s,p}$ in terms of lifting[8]

Theorem 3. *Assume that* $0 < s < 1$ *and* $1 \leq sp < N$.
a) If $sp > 1$, *then* $X^{s,p} = \{e^{\imath\varphi} \ ; \ \varphi \in W^{s,p} + W^{1,sp}\}$.
b) If $sp = 1$, *then* $X^{s,p} = W^{s,p}(Q;\mathbb{S}^1) \cap \{e^{\imath\varphi} \ ; \ \varphi \in W^{s,p} + W^{1,1}\}$.

Proof. "$\subset$" follows from Theorem 1.
"$\supset$" Let $u = e^{\imath(\varphi_1+\varphi_2)} \in W^{s,p}(Q;\mathbb{S}^1)$, with $\varphi_1 \in W^{s,p}$ and $\varphi_2 \in W^{1,sp}$. Set $u_j := e^{\imath\varphi_j}$. By Corollary 1, $u_1 \in X^{s,p}$. On the other hand, $W^{s,p}(Q;\mathbb{S}^1)$ is a group, so that $u_2 \in W^{s,p}(Q;\mathbb{S}^1)$. By Proposition 2, $u_2 \in X^{s,p}$. By (27), $X^{s,p}$ is a group, so that $u = u_1u_2 \in X^{s,p}$. □

References

[1] G. Alberti, S. Baldo, G. Orlandi, *Functions with prescribed singularities*, J. Eur. Math. Soc. **5** (2003), 275–311.
[2] R.A. Adams, Sobolev spaces, Academic Press, Pure and Applied Mathematics, 1978.
[3] J. Bourgain, H. Brezis, *On the equation div $Y = f$ and application to control of phases*, J. Amer. Math. Soc. **16** (2003), 393–426.
[4] J. Bourgain, H. Brezis, P. Mironescu, *Lifting in Sobolev spaces*, J. Anal. Math. **80** (2000), 37–86.
[5] J. Bourgain, H. Brezis, P. Mironescu, *Lifting, degree, and distributional Jacobian revisited*, Commun. Pure Appl. Math. **58** (2005), 529–551.
[6] P. Bousquet, *Topological singularities in $W^{s,p}(\mathbb{S}^N, \mathbb{S}^1)$*, Journal d'Analyse Mathématique **102** (2007), 311–346.
[7] H. Brezis, P. Mironescu, *Gagliardo-Nirenberg, composition and products in fractional Sobolev spaces*, Journal of Evolution Equations **1** (2001), 387–404.
[8] H. Brezis, P. Mironescu, *On some questions of topology for S^1-valued fractional Sobolev spaces*, Rev. R. Acad. Cien., Serie A Mat. **95** (2001), 121–143.
[9] H. Brezis, P. Mironescu, *Density in $W^{s,p}$*, in preparation.
[10] J.-Y. Chemin, Fluides parfaits incompressibles, Astérisque **230** (1995).
[11] V. Maz'ya, T. Shaposhnikova, *An elementary proof of the Brezis and Mironescu theorem on the composition operator in fractional Sobolev spaces*, Journal of Evolution Equations **2** (2002), 113–125 .
[12] P. Mironescu, *Sobolev maps on manifolds: degree, approximation, lifting*, in Perspectives in Nonlinear Partial Differential Equations, In honor of Haïm Brezis (H. Berestycki, M. Bertsch, F. Browder, L. Nirenberg, L. A. Peletier, L. Véron eds.), Contemporary Mathematics, Amer. Math. Society **446** (2007), 413–436.
[13] P. Mironescu, *Lifting default for $\mathbb{S}^1$-valued maps*, to appear in C. R. Acad. Sci. Paris **346** (2008).
[14] H.-M. Nguyen, *Inequalities related to lifting and applications*, to appear in C. R. Acad. Sci. Paris **346** (2008).
[15] A. Ponce, personal communication.
[16] R. Schoen, K. Uhlenbeck, *A regularity theory for harmonic maps*, J. Differential Geom. **17** (1982), 307–335.
[17] E. Stein, Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals, Princeton Mathematical Series, 1993.
[18] H. Triebel, Theory of function spaces, Birkhäuser Verlag, 1983.

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[8]There is another description available, in terms of distributional jacobian $T(u)$ (for its definition, see [1],[5], [6]). For $1 \leq sp < 2$, we have $X^{s,p} = \{u \in W^{s,p}(Q;\mathbb{S}^1) \ ; \ T(u) = 0\}$, result due to Bousquet [6] when $s \geq 1$ and Ponce [15] when $s < 1$.